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Converging spherical and cylindrical shock waves

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Abstract. The self-similar solutions for converging spherical and cylindrical strong shock waves in a non-ideal gas satisfying the equation of state of the Mie–Gruneisen type are investigated. The equations governing the flow, which are highly non-linear hyperbolic partial differential equations, are first reduced to a Poincaré-type ordinary differential equation with suitable approximation. Such an approximation helps in obtaining the self-similar solutions and the similarity exponent numerically by phase-plane analysis.

1. Introduction

The study of converging spherical and cylindrical shock waves is of importance because of its applications in the field of nuclear engineering (controlled thermonuclear fusion), cavitation and blast waves. The motion of converging spherical and cylindrical shock waves in a perfect gas has been studied by many authors [1–8]. The spherical shock wave analysed by Guderley [1] is one of the first examples of a class of solutions known as self-similar solutions of the second kind. Guderley [1] demonstrated the existence of the self-similar solutions for a shock wave propagating in the vicinity of the centre of convergence. Stanyukovich [2] first developed an approximate method for obtaining the similarity exponent analytically. Mishkin and Fujimoto [8], while analysing converging cylindrical and spherical shocks, observed the sudden jump in the pressure at the shock front, which continues to increase and reaches a maximum behind the shock front. This concept of single maximum pressure behind the shock front leads to an analytical determination of the shock-decay coefficient in closed form.

The converging wave is generated either by a spherical piston moving into the gas, imparting to it a certain amount of energy, or by an instantaneous release of energy on a rigid cylindrical wall, respectively. A possible method of achieving high degrees of compression is to launch successive shock waves from the ablating spherical surface. By this method one can produce a sequence of shocks of increasing strength, such that the successive shocks do not overtake each other before converging to the centre. As the wave converges to the origin, the energy becomes concentrated at the front, and the wave becomes strong and will be adiabatic until all the shocks arrive simultaneously at the point of convergence. It is assumed that the limiting motion will be self-similar as the wave converges to the center. Only a small quantity of total energy is concentrated, and it also decreases. In addition, the energy in the self-similar region decreases with time, following a power law.

The self-similar solutions are investigated for the converging cylindrical and spherical strong shock waves in a non-ideal gas (real gases) such as Helium, Nitrogen, satisfying the equation of state of the Mie–Grüneisen type. The shock is assumed to be strong and to obey a power law $x_1(t) = A(-t)^{\alpha}$ where x_1 is the shock position at the time t and A and α are

constants. The converging wave problem is studied by adopting an appropriate approximation on the Grüneisen coefficient. Such approximation alone helps in reducing the system of nonlinear differential equations to a first-order ordinary differential equation of Poincaré type. The self-similar solutions and the similarity exponent α are obtained numerically by phase-plane analysis [7]. Determination of the similarity exponent is an eigenvalue problem.

2. Basic equations

The basic conservation equations of mass, momentum and energy governing adiabatic flow in Eulerian co-ordinates are

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial r} + (\nu - 1)\frac{\rho u}{r} = 0 , \tag{2.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0 , \tag{2.2}$$

$$\frac{\partial e}{\partial t} + u\frac{\partial e}{\partial r} - \frac{p}{\rho}\left[\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right]\ln(\rho/\rho_0) = 0 , \tag{2.3}$$

where $\nu = 2$ and 3 denote the cylindrical and spherical geometries of the shock wave; ρ, u and p are density, velocity and pressure of the medium. Here ρ_0 is the initial density. The medium of flow is assumed to be obeying the equation of state of the Mie–Grüneisen type

$$p = \rho e \Gamma(\rho/\rho_0) , \tag{2.4}$$

where e is the specific internal energy, $\Gamma(\rho/\rho_0)$ is the Grüneisen coefficient. The Grüneisen coefficient for a perfect gas is a constant and is equal to $(\gamma - 1)$ where γ is the ratio of specific heats. At the shock front, $r = x_1(t)$, the boundary conditions are given by the Hugoniot jump conditions

$$\begin{aligned} &u_1 = (1 - \rho_0/\rho_1)\dot{x}_1 , \qquad p_1 - p_0 = \rho_0 u_1 \dot{x}_1 , \\ &e_1 - e_0 = \frac{1}{2}u_1^2 + \frac{p_0}{\rho_1}\left(1 - \frac{\rho_0}{\rho_1}\right) , \end{aligned} \tag{2.5}$$

where the suffixes 1 and 0 denote the quantities just behind and ahead of the shock front respectively. The dot denotes differentiation with respect to time. The strong shock conditions for the present problem can be written as

$$\begin{aligned} &u_1 = (1 - \beta)\dot{x}_1 , \\ &e_1 = p_1(1 - \beta)/(2\rho_0) , \\ &\rho_1 = \rho_0/\beta , \\ &p_1 = (1 - \beta)\rho_0 \dot{x}_1^2 , \end{aligned} \tag{2.6}$$

where β is the shock density ratio which is obtained from the strong shock conditions and equation of state (2.4), given by the relation

$$\Gamma(1/\beta)(1-\beta) = 2\beta \,. \tag{2.7}$$

The governing equations (2.1–2.3) admit several transformation groups. The suitable similarity transformations are

$$\begin{aligned} u &= v(\lambda)\dot{x}_1 \,, \\ \rho &= \rho_0 g(\lambda) \,, \\ p &= \rho_0 \dot{x}_1^2 \Pi(\lambda) \,, \end{aligned} \tag{2.8}$$

where

$$\lambda = \frac{r}{x_1}, \qquad x_1 = A(-t)^{\alpha} \,.$$

Along with the above transformations consider the set of transformations for convenience,

$$\begin{aligned} P(\lambda) &= \alpha^2 \frac{\Pi(\lambda)}{\lambda^2} \,, \\ G(\lambda) &= g(\lambda) \,, \\ U(\lambda) &= \alpha \frac{v(\lambda)}{\lambda} \end{aligned} \tag{2.9}$$

and

$$Y = P(\lambda)/G(\lambda) \,.$$

With the aid of the similarity transformations (2.8 and 2.9), the differential equations (2.1–2.3) are transformed into a system of ordinary differential equations. Thus, the transformed conservation equations of mass, momentum and energy, governing the adiabatic flow are

$$\begin{aligned} &(U-\alpha)\frac{\mathrm{d}\ln G}{\mathrm{d}\ln\lambda} + \frac{\mathrm{d}U}{\mathrm{d}\ln\lambda} + \nu U = 0 \\ &Y\frac{\mathrm{d}\ln G}{\mathrm{d}\ln\lambda} + (U-\alpha)\frac{\mathrm{d}U}{\mathrm{d}\ln\lambda} + \frac{\mathrm{d}Y}{\mathrm{d}\ln\lambda} + 2Y + U(U-1) = 0 \,, \\ &\frac{\mathrm{d}Y}{\mathrm{d}\ln\lambda} + Y\varphi(G)\frac{\mathrm{d}\ln G}{\mathrm{d}\ln\lambda} + \frac{2(U-1)Y}{U-\alpha} = 0 \,, \end{aligned} \tag{2.10}$$

where

$$\varphi = -\Gamma(G) - \frac{\mathrm{d}\ln\Gamma(G)}{\mathrm{d}\ln G} \,. \tag{2.11}$$

The transformed shock conditions are obtained from (2.6) as

$$\begin{aligned} U(1) &= (1-\beta)\alpha \,, \\ G(1) &= 1/\beta \,, \\ Y(1) &= \beta(1-\beta)\alpha^2 \,. \end{aligned} \tag{2.12}$$

The equations (2.10) governing the flow are nonlinear ordinary differential equations, and analytical solutions for such a system of equations are not possible. These equations are therefore solved numerically. Thus, the problem of obtaining the similarity exponent α of the converging cylindrical or spherical shock for the known value of the shock density ratio β, which is an eigenvalue problem, is reduced to the task of determining the integral curve on an appropriate phase-plane.

3. Numerical method

In this section we shall show the method of determining the similarity exponent using a phase-plane analysis. The transformed set of equations can be written as

$$\begin{bmatrix} 1 & (U-\alpha) & 0 \\ (U-\alpha) & Y & 1 \\ 0 & Y\varphi(G) & 1 \end{bmatrix} \begin{bmatrix} \mathrm{d}U/\mathrm{d}\ln\lambda \\ \mathrm{d}\ln G/\mathrm{d}\ln\lambda \\ \mathrm{d}Y/\mathrm{d}\ln\lambda \end{bmatrix} = \begin{bmatrix} -\nu U \\ -[2Y+U(U-1)] \\ -2\dfrac{(U-1)}{(U-\alpha)}Y \end{bmatrix} \tag{3.1}$$

and are solved for the derivatives, $\mathrm{d}U/\mathrm{d}\ln\lambda$, $\mathrm{d}\ln G/\mathrm{d}\ln\lambda$, $\mathrm{d}Y/\mathrm{d}\ln\lambda$. Thus, the above system of equations can be written, using the determinants

$$\frac{\mathrm{d}U}{\mathrm{d}\ln\lambda}=\frac{\Delta_1}{\Delta}, \qquad \frac{\mathrm{d}\ln G}{\mathrm{d}\ln\lambda}=\frac{\Delta_2}{\Delta}, \qquad \frac{\mathrm{d}Y}{\mathrm{d}\ln\lambda}=\frac{\Delta_3}{\Delta}, \tag{3.2}$$

where Δ, Δ_1, Δ_2 and Δ_3 are given as,

$$\Delta = -Y[1-\varphi(G)] + (U-\alpha)^2 ,$$

$$\Delta_1 = \nu UY[1-\varphi(G)] - U(U-1)(U-\alpha) - 2Y(1-\alpha) ,$$

$$\Delta_2 = [U(U-1)+2Y] - 2\frac{(U-1)}{(U-\alpha)}Y - \nu U(U-\alpha) ,$$

$$\Delta_3 = 2Y^2\frac{(U-1)}{(U-\alpha)} - Y\varphi(G)[U(U-1)+2Y] - (U-\alpha)[2Y(U-1) - \nu UY\varphi(G)] . \tag{3.3}$$

The coefficients of the derivatives of the system of equations (2.10) depend on U, G and Y and so do the determinants Δ, Δ_1, Δ_2 and Δ_3. In order to apply the phase-plane analysis by these equations can be solved numerically, the differential equations and the determinants must be functions of (U, Y) only. Hence, consider the approximation

$$[1-\varphi(G)] = k , \tag{3.4}$$

where the constant k depends on the material considered [9]. With this substitution, the equations (3.3) can be written as

$$\Delta = -Yk + (U-\alpha)^2 ,$$

$$\Delta_1 = \nu kUY - U(U-1)(U-\alpha) - 2Y(1-\alpha) ,$$

$$\Delta_2 = [U(U-1)+2Y] - \frac{2(U-1)}{(U-\lambda)} Y - \nu U(U-\alpha)\,,$$

$$\Delta_3 = 2Y^2 \frac{(U-1)}{(U-\alpha)} - Y(1-k)[U(U-1)+2Y] - A_1\,, \tag{3.5}$$

where

$$A_1 = (U-\alpha)[2Y(U-1) - \nu UY(1-k)]\,.$$

Dividing the third equation of (3.2) by the first we obtain a first order ordinary differential equation of Poincaré type in the (U, Y) phase-plane

$$\frac{dY}{dU} = \frac{\Delta_3(Y,U)}{\Delta_1(Y,U)} = \frac{Y}{(U-\alpha)} \frac{M(Y,U)}{N(Y,U)}\,, \tag{3.6}$$

where

$$M(Y,U) = [2Y(U-1) - (1-k)(U-\alpha)[U(U-1)+2Y] - (U-\alpha)^2[2(U-1) - \nu U(1-k)]]\,, \tag{3.7}$$

$$N(Y,U) = [\nu UYk - U(U-1)(U-\alpha) - 2Y(1-\alpha)]\,. \tag{3.8}$$

Thus, the above equation (3.6) is to be solved numerically, subject to the boundary conditions:
on the image of the shock front:

$$U(1) = (1-\beta)\alpha, \qquad Y(1) = \beta(1-\beta)\alpha^2\,, \tag{3.9}$$

and the image of the free boundary, a singularity of saddle type of (3.6):

$$U \equiv \mu = \frac{2(1-\alpha)}{\nu k}\,, \qquad Y = \infty\,. \tag{3.10}$$

With respect to these types of boundary condition, equation (3.6) is undefined and hence we introduce a new variable $X = V(\lambda)/P(\lambda)$, so that the boundary conditions (3.9) and (3.10) can be written as

$$U = (1-\beta)\alpha\,, \qquad X = 1/[\beta(1-\beta)\alpha^2] \tag{3.11}$$

$$U \equiv \mu = \frac{2(1-\alpha)}{\nu k}\,, \qquad X = 0\,. \tag{3.12}$$

The variable λ must increase monotonically upon moving from the image of the free boundary to the shock front. In general, it is impossible to satisfy these conditions for any arbitrary value of α, since one of them is a saddle type singularity. However, there exists one exceptional value of α for which this is possible. Thus, the problem reduces to a nonlinear eigenvalue problem. Equation (3.6) is solved in obtaining the integral curve in the (U, X) phase-plane. Each of these integral curves is associated with the saddle-type singularity, corresponding to the centre of symmetry, given by $X = 0$ and $U = \mu$. Further, the singularity

depends on α and k. The slope of the integral curve at the saddle type singular point is obtained from equation (3.6) by means of de l'Hôpital's rule, given by,

$$\frac{dX}{dU} = \frac{2(\alpha - 1) + (\mu - \alpha)2k}{\mu(\mu - 1)(\mu - \alpha)^2}. \tag{3.13}$$

The simplification performed on the differential equation is insufficient to start the numerical procedure, since the Grüneisen coefficient is unknown. Hence, equation (3.4) can be written as

$$1 + \Gamma(G) + \frac{d \ln \Gamma(G)}{d \ln G} = k, \quad k > 0. \tag{3.14}$$

Integrating the equation (3.14), we obtain the Grüneisen coefficient

$$\Gamma(G) = \frac{(1 - k)\Gamma}{(1 - k + \Gamma)G^{(1-k)} - \Gamma}, \quad \Gamma = \Gamma(1). \tag{3.15}$$

Thus, for the values of $k = 1.42$, $\Gamma = 2.12$ and $k = 1.298$, $\Gamma = 1.017$ (see Neal [9]) of Aluminum and Lithium, equation (3.15) yields

$$\Gamma(G) = \frac{0.42\Gamma}{\Gamma - (\Gamma - 0.42)G^{-0.42}}, \tag{3.16}$$

$$\Gamma(G) = \frac{0.298\Gamma}{\Gamma - (\Gamma - 0.298)G^{-0.298}}, \tag{3.17}$$

respectively. The value of β for each of these Grüneisen coefficients is obtained from the strong shock condition (2.7). After obtaining the Grüneisen coefficients for different materials in condensed matter, the system of equations is reduced to one of Poincaré type in a (U, X) phase-plane. Thus, using the phase-plane analysis, we solve equation (3.6) numerically from the singular point of saddle type. The desired solution must pass through

Table 1. Similarity exponents for perfect gas

	Cylindrical geometry	Spherical geometry
γ	α	α
1.1	0.885249	0.795969
1.4	0.835323	0.717174
5/3	0.815625	0.688377
3	0.775667	0.636411

Table 2. Similarity exponents for Aluminium and Lithium

	Cylindrical symmetry ($\nu = 2$)		Spherical symmetry ($\nu = 3$)	
	Aluminium	Lithium	Aluminium	Lithium
Γ	2.12	1.017	2.12	1.017
k	1.42	1.298	1.42	1.298
β	0.4571123	0.2155141	0.4571123	0.2155141
α	0.681796988	0.54385697	0.70864358	0.71427626

the two singular points. Using the values of k, Γ, and β, along with the Gruneisen coefficients for Aluminium and Lithium, the unique value of the similarity exponent α is calculated. These values of α, along with the perfect-gas values, are shown in Tables 1 and 2.

4. Concluding remarks

The similarity exponents are obtained for the spherical and cylindrical converging shock waves propagating in a non-ideal media. The type of assumption on the Grüneisen coefficient considered alone makes the problem solvable, using a phase-plane analysis. The numerical method discussed is useful in determining the similarity exponent α for a class of problems using different Grüneisen coefficients. Equation (2.4) represents the general form of the equation of state. For different values of k, equation (3.14) represents different equations of state of the Mie–Grüneisen type. The perfect-gas solutions can be recovered by substituting $k = \gamma$ and $\Gamma = (\gamma - 1)$ in equation (3.14).

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